Lecture 15: Exact Homology Sequence

Exact Homology Sequence

Definition

Chain maps $0 \to C'_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} C''_\bullet \to 0$ is called a short exact sequence if for each n

$$0 \to C'_n \xrightarrow{i} C_n \xrightarrow{p} C''_n \to 0$$

is an exact sequence of abelian groups.

We have the following commuting diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C'_{n+1} \arrow[r, "i"] \arrow[d, "\partial'"] & C_{n+1} \arrow[r, "p"] \arrow[d, "\partial"] & C''_{n+1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_{n} \arrow[r, "i"] \arrow[d, "\partial'"] & C_{n} \arrow[r, "p"] \arrow[d, "\partial"] & C''_{n} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i"] \arrow[d, "\partial'"] & C_{n-1} \arrow[r, "p"] \arrow[d, "\partial"] & C''_{n-1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Lemma

Let $0 \to C'_\bullet \xrightarrow{i} C_\bullet \xrightarrow{p} C''_\bullet \to 0$ be a short exact sequence. There is a natural homomorphism

$$\delta : H_n(C''_\bullet) \to H_{n-1}(C'_\bullet)$$

called the connecting map. It induces a long exact sequence

$$\cdots \to H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} \cdots$$

The connecting map δ is natural in the sense that a commutative diagram of complexes with exact rows

$$\begin{tikzcd} 0 \arrow[r] & C'_\bullet \arrow[r] \arrow[d] & C_\bullet \arrow[r] \arrow[d] & C''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D'_\bullet \arrow[r] & D_\bullet \arrow[r] & D''_\bullet \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram of abelian groups with exact rows

$$\begin{tikzcd} \cdots \arrow[r] & H_n(C''_\bullet) \arrow[r] \arrow[d] & H_n(C_\bullet) \arrow[r] \arrow[d] & H_n(C''_\bullet) \arrow[r, "\delta"] \arrow[d] & H_{n-1}(C'_\bullet) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_n(D''_\bullet) \arrow[r] & H_n(D_\bullet) \arrow[r] & H_n(D''_\bullet) \arrow[r, "\delta"] & H_{n-1}(D'_\bullet) \arrow[r] & \cdots \end{tikzcd}$$

Proof

We first describe the construction of δ. Given a class $[\alpha] \in H_n(C''_\bullet)$, let $\alpha \in C''_n$ be a representative. We can find $\beta \in C_n$

$$p(\beta) = \alpha.$$

Since $p(\partial\beta) = \partial(p(\beta)) = \partial\alpha = 0$, we find $\gamma \in C'_{n-1}$ such that

$$i(\gamma) = \partial\beta.$$

Since

$$i(\partial(\gamma)) = \partial(i(\gamma)) = \partial^2(\beta) = 0 \Longrightarrow \partial(\gamma) = 0.$$

This is illustrated by chasing the following diagram

$$\begin{tikzcd}
 & \beta \arrow[r] \arrow[d] & \alpha \arrow[d] \\
\gamma \arrow[r] \arrow[d] & \partial\beta \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

γ defines a class $[\gamma] \in H_{n-1}(C'_\bullet)$. We show this class does not depend on various choices.

- Choice of β. Suppose we choose another $\tilde{\beta}$ such that $p(\tilde{\beta}) = \alpha$. Then there exists $x \in C'_n$ such that

$$\tilde{\beta} = \beta + i(x).$$

  It follows that $\tilde{\gamma} = \gamma + \partial x$, so $[\tilde{\gamma}] = [\gamma]$.

- Choice of α. Suppose we choose another representative $\tilde{\alpha} = \alpha + \partial x$ of the class $[\alpha]$. We can choose a lifting

$$\tilde{\beta} = \beta + \partial y$$

  of $\tilde{\alpha}$ where $p(y) = x$. Since $\partial\tilde{\beta} = \partial\beta$, we have $\tilde{\gamma} = \gamma$.

Therefore we have a well-defined map

$$\delta : H_n(C''_\bullet) \to H_{n-1}(C'_\bullet)$$

by

$$\delta[\alpha] = [\gamma].$$

We next show the exactness of the sequence

$$\cdots \to H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} \cdots$$

$$\cdots \to H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} \cdots$$

Exactness at $H_n(C_\bullet)$

$\mathrm{im}(i_*) \subset \ker(p_*)$ is obvious.

If $[\alpha] \in H_n(C_\bullet)$ such that $[p(\alpha)] = 0$, so $p(\alpha) = \partial x$. Let $y \in C_{n+1}$ be a lifting of x so $p(y) = x$. Since $p(\alpha - \partial y) = 0$,

$$\alpha - \partial y = i(\beta)$$

for some $\beta \in C'_n$. Then $\partial\beta = 0$ and

$$i_*([\beta]) = [\alpha]$$

which implies $\ker(p_*) \subset \mathrm{im}(i_*)$.

$$\cdots \to H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} \cdots$$

Exactness at $H_{n-1}(C''_\bullet)$

Assume $[\alpha] = p_*[\beta]$, then β is a lift of α and $\partial\beta = 0$. So $\delta[\alpha] = 0$.

$$\mathrm{im}(p_*) \subset \ker(\delta).$$

On the other hand, if $\delta[\alpha] = 0$. We can find a lift β of α such that $\partial\beta = 0$. Then $[\alpha] = p_*[\beta]$. Hence

$$\ker(\delta) \subset \mathrm{im}(p_*).$$

$$\cdots \to H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C'_\bullet) \xrightarrow{i_*} H_{n-1}(C_\bullet) \xrightarrow{p_*} \cdots$$

Exactness at $H_{n-1}(C'_\bullet)$

$i_*\delta([\alpha]) = i_*[\gamma] = [\partial\beta] = 0$. This shows

$$\operatorname{im}\delta \subset \ker i_*.$$

Assume $[\gamma] \in H_{n-1}(C'_\bullet)$ such that $i_*[\gamma] = 0$. Then $i(\gamma) = \partial\beta$ for some β. Let $\alpha = p(\beta)$. Then

$$\partial(\alpha) = \partial p(\beta) = p(\partial\beta) = pi(\gamma) = 0.$$

So $[\alpha]$ defines a homology class and $\delta[\alpha] = [\gamma]$ by construction.

$$\ker i_* \subset \operatorname{im}\delta.$$

$\square$

Relative Homology

Definition

Let $A \subset X$ be a subspace. It indues a natural injective chain map $S_\bullet(A) \hookrightarrow S_\bullet(X)$. We define the singular chain complex of X relative to A to be

$$S_n(X, A) := S_n(X)/S_n(A)$$

with the induced differential. Its homology

$$\mathrm{H}_n(X, A) := \mathrm{H}_n(S_\bullet(X, A))$$

is called the *n-th relative homology*.

Theorem

For $A \subset X$, there is a long exact sequence of abelian groups

$$\cdots \to \mathrm{H}_n(A) \to \mathrm{H}_n(X) \to \mathrm{H}_n(X, A) \xrightarrow{\delta} \mathrm{H}_{n-1}(A) \to \cdots$$

Proof.

This follows from the short exact sequence of complexes

$$0 \to S_\bullet(A) \to S_\bullet(X) \to S_\bullet(X, A) \to 0.$$

□

Define relative n-cycles $Z_n(X, A)$ and relative n-boundaries $B_n(X, A)$

$$Z_n(X, A) = \{\gamma \in S_n(X) : \partial\gamma \in S_{n-1}(A)\}$$
$$B_n(X, A) = B_n(X) + S_n(A) \subset S_n(X).$$

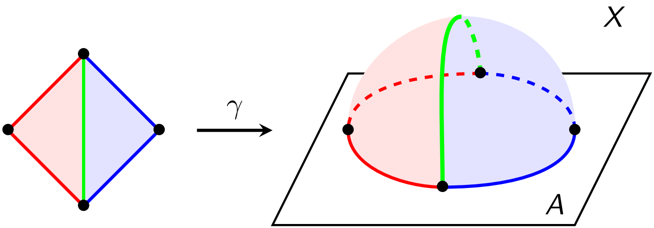


图: Elements in $Z_n(X, A)$

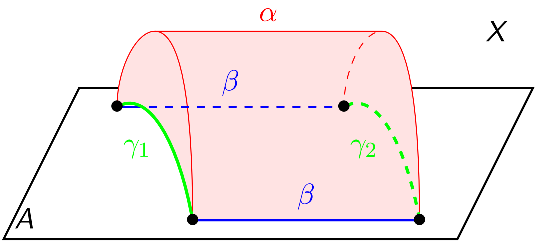
Then it is easy to check that

$$S_n(A) \subset B_n(X, A) \subset Z_n(X, A) \subset S_n(X)$$

and

$$\mathrm{H}_n(X, A) = Z_n(X, A)/B_n(X, A).$$

Two relative n-cycles γ_1, γ_2 defines the same class $[\gamma_1] = [\gamma_2]$ in $\mathrm{H}_n(X, A)$ if and only if $\gamma_1 - \gamma_2$ is homologous to a chain in A.



$$\partial\alpha = \beta + \gamma_1 - \gamma_2,\ \beta \in S_\bullet(A)$$
$$\Longrightarrow [\gamma_1] = [\gamma_2] \quad \in H_n(X, A)$$

图: Relative n-cycles

The connecting map

$$\delta : \mathrm{H}_n(X, A) \to \mathrm{H}_{n-1}(A)$$

can be understood as follows: an n-cycle in $\mathrm{H}_n(X, A)$ is represented by an n-chain $\gamma \in S_n(X)$ such that its boundary $\partial(\gamma)$ lies in A. Viewing $\partial(\gamma)$ as a $(n-1)$-cycle in A, then

$$\delta[\gamma] = [\partial(\gamma)].$$

Let $f\colon (X,A) \to (Y,B)$ be a map of pairs. It naturally induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A) \arrow[r] \arrow[d] & S_\bullet(X) \arrow[r] \arrow[d] & S_\bullet(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_\bullet(B) \arrow[r] & S_\bullet(Y) \arrow[r] & S_\bullet(Y,B) \arrow[r] & 0
\end{tikzcd}$$

which further induces compatible maps on various homology groups

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_n(A) \arrow[r] \arrow[d] & \mathrm{H}_n(X) \arrow[r] \arrow[d] & \mathrm{H}_n(X,A) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_n(B) \arrow[r] & \mathrm{H}_n(Y) \arrow[r] & \mathrm{H}_n(Y,B) \arrow[r, "\delta"] & \mathrm{H}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

This construction can be generalized to the triple $B \subset A \subset X$.

Theorem

If $B \subset A \subset X$ are subspaces, then there is a long exact sequence

$$\cdots \to \mathrm{H}_n(A, B) \to \mathrm{H}_n(X, B) \to \mathrm{H}_n(X, A) \xrightarrow{\delta} \mathrm{H}_{n-1}(A, B) \to \cdots .$$

Proof.

This follows from the long exact sequence associated to

$$0 \to \frac{S_\bullet(A)}{S_\bullet(B)} \to \frac{S_\bullet(X)}{S_\bullet(B)} \to \frac{S_\bullet(X)}{S_\bullet(A)} \to 0.$$

$\square$

Theorem (Homotopy Axiom for Pairs)

If $f, g : (X, A) \to (Y, B)$ and f is homotopic to $g \ \mathrm{rel}\, A$. Then

$$\mathrm{H}_n(f) = \mathrm{H}_n(g) : \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B).$$

Reduced Homology

Proposition

Let $\{X_\alpha\}$ be path connected components of X, then

$$\mathrm{H}_n(X) = \bigoplus_\alpha \mathrm{H}_n(X_\alpha).$$

Proof.

This is because

$$S_\bullet(X) = \bigoplus_\alpha S_\bullet(X_\alpha).$$

$\square$

Proposition

Let X be path connected. Then $\mathrm{H}_0(X) \simeq \mathbb{Z}$.

Proof.

$\mathrm{H}_0(X) = S_0(X)/\partial S_1(X)$. Let us define the map

$$\epsilon : S_0(X) \to \mathbb{Z}, \quad \sum_{p \in X} m_p p \to \sum_p m_p.$$

ϵ is zero on $\partial S_1(X)$.
On the other hand, assume $\epsilon(\sum_{p \in X} m_p p) = 0$, then we can write

$$\sum_{p \in X} m_p p = \sum_i (p_i - q_i)$$

into pairs. Since X is path connected, we can find a path $\gamma_i : I \to X$ such that $\partial \gamma = p_i - q_i$. Therefore $\sum_{p \in X} m_p p = \sum_i \partial \gamma_i \in \partial S_1(X)$.
It follows that ϵ induces an isomorphism $\epsilon : \mathrm{H}_0(X) \simeq \mathbb{Z}$.

In general, we have a surjective map

$$\epsilon : \mathrm{H}_0(X) \to \mathbb{Z}, \quad \sum_{p \in X} m_p p \to \sum_p m_p.$$

Definition

We define the reduced homology group of X by

$$\tilde{\mathrm{H}}_n(X) = \begin{cases} \mathrm{H}_n(X) & n > 0 \\ \ker(\mathrm{H}_0(X) \to \mathbb{Z}) & n = 0 \end{cases}$$

We can think about the reduced homology group as the homology group of the chain complex

$$\cdots \to S_2(X) \to S_1(X) \to S_0(X) \to \mathbb{Z}.$$

The long exact sequence still holds for the reduced case

$$\cdots \to \tilde{\mathrm{H}}_n(A) \to \tilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X, A) \xrightarrow{\delta} \tilde{\mathrm{H}}_{n-1}(A) \to \cdots$$

Example

If X is contractible, then $\tilde{H}_n(X) = 0$ for all n.

Example

Let $x_0 \in X$ be a point. Using the long exact sequence for $A = \{x_0\} \subset X$, we find

$$\mathrm{H}_n(X, x_0) = \tilde{\mathrm{H}}_n(X).$$